P.Grinevich, A.Mironov, S.Novikov[1]

2D Schrodinger Operator, (2+1) Systems and New Reductions. The 2D Burgers Hierarchy and Inverse Problem Data[2]

Abstract. The Theory of (2+1) Systems based on 2D Schrodinger Operator was started by S.Manakov, B.Dubrovin, I.Krichever and S.Novikov in 1976 (see[1,2]). The Analog of Lax Pairs introduced in [1], has a form $L_t = [L, H] - fL$ (”The L, H, f-triples”) where $L = \partial_x \partial_y + G\partial_y + S$ and H, f-some linear PDEs. Their Algebro-Geometric Solutions and therefore the full higher order hierarchies were constructed in [2]. The Theory of 2D Inverse Spectral Problems for the Elliptic Operator L with x, y replaced by $z, \bar{z}$, was started in [2]: The Inverse Spectral Problem Data are taken from the complex ”Fermi-Curve” consisting of all Bloch-Floquet Eigenfunctions $L\psi = const$. Many interesting systems were found later [3]. However, specific properties of the very first system offered in [1] for the verification of new method only, were not studied more than 10 years until B.Konopelchenko found in 1988 (see [6]) analogs of Backund Transformations for it. He pointed out on the “Burgers-Type Reduction”[3]. Indeed, the present authors quite recently found very interesting extensions, reductions and applications of that system both in the theory of nonlinear evolution systems (The Self-Adjoint and 2D Burgers Hierarhies were invented, and corresponding reductions of Inverse Problem Data found) and in the Spectral Theory of Important Physical Operators (”The Purely Magnetic 2D Pauli Operators”). We call this system GKMMN by the names of authors who studied it.

Let us consider the 2nd order operators L, H and scalar function f, re-

[1]All Novikov’s works quoted here, can be taken directly from his homepage http://www.mi.ras.ru/~snovikov, click **Publications**. Our e-mails are pgg@landau.ac.ru (Grinevich), mironov@nsc.math.ru (Mironov), novikov@ipst.umd.edu (Novikov)

[2]This work was accepted for publication in the Journal Russian Math Surveys. It will appear in 2010, v 65 n 3

[3]Unfortunately, in the work [6] this system is presented as a new one. At the same time, the work [1] where it was originally found, is included in the list of references of [6]. The work [2] where Algebro-Geometric Solutions of such systems were found in 1976, is not quoted at all in [6]. It leads to the wrong impression that there was no development till 1980, reversing the chronological order of participating authors to the opposite

duced to the following form by the gauge transformations

$$L = \partial_x \partial_y + G\partial_y + S, H = \partial_x^2 + F\partial_y + A$$

Using L, H, f-triple (see Abstract) we define corresponding (2+1)nonlinear evolution system. We call it "The GKMMN System". Making calculation, we obtain following

Proposition. The GKMMN System has a form(I)

$$G_t = G_{xx} - G_{yy} + (F^2)_x - (G^2)_x - A_x + 2S_y, S_t = -S_{xx} + S_{yy} + 2(GS)_x - 2(FS)_y$$

$$F_x = 2G_y, A_y = 2S_x, f = 2G_x - F_y$$

Let us formulate some useful corollaries of that system.

Corollary 1. The system GKMMN is compatible with the purely real reduction where all coefficients are real.

Corollary 2. The system GKMMN admits a Reduction $S = 0$. We call it **"The 2D Burgers System"** and denote by B_2. The whole Hierarchy can be naturally constructed after Theorem 2 below.

Corollary 3. For the GKMMN system and its stationary problem the elliptic operator H can be self-adjoint only in the trivial cases reducible to the functions of one variable. Here $H = \Delta + F\partial_y + A$ is such that the magnetic field $B = F_x/i, i^2 = -1$, and electric potential $U = A - F^2/4 - F_y/2$ are real and smooth.

Proof of this corollary requires calculations. Under these restrictions the system GKMMN became strongly over-determined leading to the complete degeneration.

Conjecture. For the smooth periodic second order self-adjoint elliptic 2D operators the complete complex manifold of the Bloch-Floquet Eigenfunctions W (except some trivial cases reducible to one variable), cannot contain Zariski Open Part of the Complex Algebraic Curve $\Gamma \subset W$ except of the levels $\epsilon = const$ found in 1976 in [2].

Corollary 4. The substitution

$$G = -(\log c)_x, F = -2(\log c)_y, A = -2u_x, S = -u_y$$

transforms our system into the following system (II):

$$[(c_t - c_{xx} + c_{yy})c^{-1}]_x = 2(u_{yy} - u_{xx}), u_t = u_{yy} - u_{xx} + 2(u_y c_x/c)_x - 2(u_y c_y/c)_y$$

The B_2 Reduction $S = 0$ reduces system to the linear form (III):

$$c_t - c_{xx} + c_{yy} = (U(x) + V(y))c$$

exactly in the same way as the ordinary 1D Burgers System (i.e. our system with $U = V = 0$ not depending on the variable y).

The spectral meaning of this variables and substitution will be clarified below for the Algebro-Geometric (AG) Solutions immediately leading to the full Hierarchy of such systems.

The Algebro-Geometric (AG) Inverse Spectral Problem Data.

Take Riemann Surface Γ with 2 "infinite" points ∞_1, ∞_2 and local parameters $1/k_1, 1/k_2$ near them, $1/k_j(\infty_j) = 0$. Select the "Divisor of poles" $D = P_1 + ... + P_g$ in Γ. Construct "The 2-point Baker-Akhiezer Function" $\psi(P, x, y, t)$ invented in [2]. It should be meromorphic in the variable $P \in \Gamma$ outside of infinities, with divisor of poles D which is x, y, t-independent. Its asymptotic behavior near infinities is following:

$$\psi = ce^{k_1x+k_1^2t}(1 + v/k_1 + O(1/k_1^2)), \psi = e^{k_2y+k_2^2t}(1 + u/k_2 + O(1/k_2^2))$$

This function satisfies to the equation $L\psi = 0$ and to the (2+1)-systems (I,II,III) with parameters (c, u) entering it.

The Real AG Reduction of (I) is following: There is an antiholomorphic involution $\sigma : \Gamma \to \Gamma, \sigma^2 = 1$, such that

$$\sigma(\infty_j) = \infty_j, \sigma^*(k_j) = -\bar{k_j}, \sigma(D) = D$$

Easy to formulate conditions such that real solutions (written through the Θ-functions) are smooth nonsingular. For the dense family of data they are periodic. In general they are quasiperiodic as usual.

The Stationary AG Solutions are such that $[L, H] = fL$ and $H\psi = \lambda(P)\psi$ where H is an elliptic operator as above. They correspond to the algebraic curves Γ with algebraic function λ having exactly 2 poles on Γ of the second order in both infinite points ∞_1, ∞_2. However, they are non-self-adjoint in the nontrivial cases.

The Burgers Reduction B_2 is especially interesting. Here $S = u_y = 0$.

Theorem 1. Take reducible Riemann Surface $\Gamma = \Gamma' \bigcup \Gamma''$ such that

$$\Gamma' \bigcap \Gamma'' = Q = Q_0 \bigcup ... \bigcup Q_k, \infty_1 \in \Gamma' \infty_2 \in \Gamma''$$

and divisor

$$D = D' + D'', D' \subset \Gamma', D'' \subset \Gamma'', |D'| = g' + k, |D''| = g''$$

where g'=genus of Γ', D''=genus of Γ'', all points ∞_j, Q, D are distinct. Construct ψ as a standard one-point Baker Akhiezer function ψ'' on Γ'' with divisor D'' and asymptotic $\psi'' = e^{k_2x+k_2^2t}(1 + O(1/k_2))$ On the part Γ' our function ψ should coincide with ψ'. It has the divisor of poles D', asymptotic $\psi' = ce^{k_1y+k_1^2t}(1 + O(1/k_1)$ and conditions (*)

$$\psi'(Q_s) = \psi''(Q_{\sigma(s)})$$

where σ is some permutation of the set Q. Then we have $L(\psi) = 0$ and $(L_t - [L, H])\psi = 0$ with $S = u_y = 0$.

Remark. We can drop the surface Γ'' and divisor D''. Take any set of solutions $\psi''_s(x, t)$ to the equation $\psi''_{s,t} = \psi''_{s,xx}, s = 0, 1, ..., k$. Define $\psi'(x, y, t, P)$ using conditions $\psi'(x, y, t, Q_s) = \psi''_s(x, t)$ instead of conditions (*). Our function $\psi = \psi'$ satisfies to the equations $L\psi = 0, L_t = [L, H] - fL$ for all points $P \in \Gamma'$, and $S = u_y = 0$.

Corresponding hierarhy with higher times we call "The 2D Burgers Hierarhy" B_2.

There are 2 cases in our theory:

1.$(x, y) \in R$. This is the system $GKMMN - I$ and reduction $B_2 - I$

2.$x \to z, y \to \bar{z}$ and $\partial_x \to \partial = \partial_x - i\partial_y, \partial_y \to \bar{\partial} = \partial_x + i\partial_y, \partial\bar{\partial} = \Delta$. This is the system $GKMMN - II$ and reduction $B_2 - II$.

Theorem 2. For the system $GKMMN - II$ in the variables $z, \bar{z}$ the reduction to the class of self-adjoint operators L with real magnetic field $-2B = 2G_{\bar{z}} = F_z$ and potential $S \in R$ is compatible with time dynamics in the variable $it, t \in R$ (V):

$$[(c_t - 4c_{xy})c^{-1}]_z = 8u_{xy}, (u + 4u_{xy})_{\bar{z}} = 2/i[(u_{\bar{z}}c_z/c) - (u_{\bar{z}}c_{\bar{z}}/c)_{\bar{z}}]$$

Here we have $S = u_{\bar{z}} \in R, c = e^{2\Phi} \in R$, and system can be written in the form (VI):

$$c_t - 4c_{xy} = 8a_yc = -4Im(u_z)c, S_t + 4S_{xy} = 8[S\Phi_{xy} - S_x\Phi_y - S_y\Phi_x]$$

with $u = a + ib, S = a_x - b_y, a_y + b_x = 0$.

The condition $S = 0$ leads to the linear system $B_2 - II$ (formula (VII)):

$$c_t - 4c_{xy} = T(x, y, t)c, \Delta T = 0, T = 8a_y \in R$$

Here $G = 1/2(\log c)_z, B = -1/2\Delta(\log c) \in R$.

For the self-adjoint factorizable operator L (i.e. $S = 0$), the Reducible Riemann Surface Γ admits an anti-involution $\sigma : \Gamma' \to \Gamma''$ and back, $\sigma^2 = 1$. **The spectrum of this operator determines the spectrum of Purely Magnetic Nonrelativistic 2D Pauli Operator for the particles with spin 1/2. The theory of ground states for the Algebro-Geometric Pauli Operators is developed in [4]**. Results of the present work (without theorem 2) can be found in our article in arXiv (see [5]).

References.

1.S.Manakov, Uspekhi Math Nauk, 1976 v 31 n 5 pp 245-246

2.B.Dubrovin, I.Krichever, S.Novikov, Doklady AN SSSR, 1976, v 229 n 1 pp 15-18

3.A.Veselov, S.Novikov, Doklady AN SSSR, 1984, v 279 v 4 pp 784-788

4.P.Grinevich, A.Mironov, S.Novikov, arXiv: 1004.1157

5.P.Grinevich, A.Mironov, S.Novikov, arXiv: 10014300

6.B.Konopelchenko, Inverse Problems, 1988, v.4, pp 151-163.